REMARK ON POLARIZED K3 SURFACES OF GENUS 36

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ABSTRACT. Smooth primitively polarized K3 surfaces of genus 36 are studied. It is proved that all such surfaces S, for which there exists an embedding $\mathrm{R} \hookrightarrow \mathrm{Pic}(S)$ of some special lattice R of rank 2, are parameterized up to an isomorphism by some 18-dimensional unirational algebraic variety. More precisely, it is shown that a general S is an anticanonical section of a (unique) Fano 3-fold with canonical Gorenstein singularities.

1. INTRODUCTION

Let $\mathcal{K}_g$ be the moduli space of all smooth primitively polarized K3 surfaces of genus g. $\mathcal{K}_g$ is known to be a quasi-projective algebraic variety (see for example [25]). This makes it possible to consider the fundamental questions of birational geometry about $\mathcal{K}_g$ such as its rationality, unirationality, rational connectedness, Kodaira dimension, and etc.

S. Mukai's vector bundle method, developed to classify higher dimensional Fano manifolds of Picard number 1 and coindex 3 (see [15], [18]), allowed to prove unirationality of $\mathcal{K}_g$ for $g \in \{2, \ldots, 10, 12, 13, 18, 20\}$ (see [17], [20], [16], [21]). At the same time, $\mathcal{K}_g$ turns out to be non-unirational for general $g \geqslant 43$ (see [4], [13], [14]). In principle, the proof of unirationality of $\mathcal{K}_g$ is based on the observation that general K3 surface S_g with primitive polarization L_g and "not very big" g is an anticanonical section of a smooth Fano 3-fold X_g of genus g so that $L_g = -K_{X_g}\big|_{S_g}$ (see [17], [16], [19]). The latter gives a rational dominant map from the moduli space $\mathcal{F}_g$ of pairs (X_g, S_g), where $S_g \in |-K_{X_g}|$ is smooth, to $\mathcal{K}_g$ by sending (X_g, S_g) to S_g, with $\mathcal{F}_g$ typically being a rational algebraic variety. However, this construction has the restriction that X_g must have Picard number 1, which does not hold for most g (see [7]).

In order to generalize the above arguments for every possible g, to a given smooth Fano 3-fold V of genus g one associates the Picard lattice $R_V := \mathrm{Pic}(V)$, equipped with the pairing $(D_1, D_2) := D_1 \cdot D_2 \cdot (-K_V)$ for D_1, $D_2 \in \mathrm{Pic}(V)$, and considers the moduli space $\mathcal{K}_g^{R_V}$ of all smooth K3 surfaces S_g, equipped with a primitive embedding $R_V \hookrightarrow \mathrm{Pic}(S_g)$ which maps $-K_V$ to an ample class on S_g of square g (let us call such S_g a K3 *surface of type* R_V). A beautiful result due to A. Beauville states that a general K3 surface of type R_V is the anticanonical section of a smooth Fano 3-fold X_g of genus g such that $R_{X_g} \simeq R_V$ (see [1]). The proof employs the same idea as above, but instead of $\mathcal{F}_g$ the moduli space $\mathcal{F}_g^{R_V}$ of pairs (X_g, S_g), where $S_g \in |-K_{X_g}|$ is smooth and X_g is equipped with the lattice isomorphism $R_{X_g} \simeq R_V$, is considered. Again the forgetful map $(X_g, S_g) \mapsto S_g$ from $\mathcal{F}_g^{R_V}$ to $\mathcal{K}_g^{R_V}$ turns out to be generically surjective. However, these arguments can be applied only to some $g \leqslant 33$ (see [7]).

In the present paper, we study primitively polarized smooth K3 surfaces of genus 36 and consider the following

Conjecture 1.1. *The moduli space $\mathcal{K}_{36}$ is unirational.*

To develop an approach to prove Conjecture 1.1, we employ the above ideas to realize a general smooth primitively polarized K3 surface of genus 36 as an anticanonical section of some Fano 3-fold, which must be singular in this case (see [7]). The natural candidate for the latter is the Fano 3-fold X with canonical Gorenstein singularities and genus 36, constructed and studied in [9], [8]. This X has only one singular point (see Corollary 3.10) and the anticanonical linear system $|-K_X|$ gives an embedding $X \hookrightarrow \mathbb{P}^{37}$ (see Remark 3.12), which implies that a general surface $S \in |-K_X|$ is smooth. Also the Picard group of X is generated by K_X (see Corollary 3.11).

Unfortunately, the divisor class group of X has two generators, K_X and some surface E (see Corollary 3.11), so that the restrictions $K_X\big|_S$ and $E\big|_S$ generate a primite sublattice R_S in $\mathrm{Pic}(S)$. In particular, the Picard number of S must be at least 2, and hence S can not be general. However, all lattices R_S, $S \in |-K_X|$, are isomorphic to the lattice $\mathrm{R} \simeq \mathbb{Z}^2$ with the associated quadratic form $70x^2 + 4xy - 2y^2$ (see the end of Section 3), and, as above,

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we can consider the moduli space $\mathcal{K}_{36}^{\mathrm{R}}$ of K3 surfaces of type R. On the other hand, we may also consider the moduli space $\mathcal{F}$ of pairs $(X^\sharp, S^\sharp)$, where $X^\sharp$ is a Fano 3-fold of genus 36 with canonical Gorenstein singularities and $S^\sharp \in |-K_{X^\sharp}|$ is smooth (see Remark 1.3 below for the precise description of $\mathcal{F}$). Let us state the main result of the present paper:

Theorem 1.2. *The forgetful map $s : \mathcal{F} \longrightarrow \mathcal{K}_{36}^{\mathrm{R}}$ is generically surjective.*

Remark 1.3. In the proof of Theorem 1.2, we do not appeal to Akizuki–Nakano Vanishing Theorem, used in [1] to show that $\mathcal{F}_g$ (or $\mathcal{F}_g^{R_V}$) is a smooth stack, since it is not clear how to apply this theorem in the singular case. Instead, we note that X is unique up to an isomorphism (see Proposition 3.7), and, moreover, it admits a crepant resolution $f : Y \longrightarrow X$, with Y being also unique up to an isomorphism (see Proposition 3.8). Then one can prove (see Proposition 4.1) that $\mathcal{F}$ carries the structure of a normal scheme, being the geometric quotient $U/\mathrm{Aut}(Y)$ of an open subset U in $\mathbb{P}^{37}$ by the group $\mathrm{Aut}(Y)$ of regular automorphisms of Y. The proof of Theorem 1.2 then goes along the same lines as in [1] (see Lemma 4.10 below).

Remark 1.4. Taking $X = \mathbb{P}(1,1,1,3)$ in the above considerations, one might apply the arguments from [1] directly (cf. Remark 1.3) to prove that the moduli space $\mathcal{K}_{10}$ is unirational (see [9], [8] for geometric properties of $\mathbb{P}(1,1,1,3)$).

Furthermore, since the forgetful map $\mathcal{K}_{36}^{\mathrm{R}} \longrightarrow \mathcal{K}_{36}$ is finite and representable (see [1, (2.5)]), from Theorem 1.2, construction of $\mathcal{F}$ and quasi-projectivity of $\mathcal{K}_{36}$ we deduce the following

Corollary 1.5. *There exists a 18-dimensional unirational algebraic variety which parameterizes up to an isomorphism all smooth* K3 *surfaces of type* R*. For general such surface S, $S \in |-K_X|$ and the Picard lattice of S is isomorphic to* R.

Remark 1.6. On the opposite, it follows from the proof of Theorem 1.2 and [2], [3], [23] that no general smooth primitively polarized K3 surface S of genus 36 can be an ample anticanonical section of a normal algebraic 3-fold, except for the cone over S.

Remark 1.7. Corollary 1.5 gives only unirational hypersurface in $\mathcal{K}_{36}$ but not the whole $\mathcal{K}_{36}$, and hence the proof of Conjecture 1.1 is still to go. It would be also interesting to know whether the map s from Theorem 1.2 is 1-to-1 and $\mathcal{K}_{36}^{\mathrm{R}}$ is rational (it follows from the proof of Theorem 1.2 that s is generically étale).

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2. Notation and conventions

We use standard notions and facts from the theory of minimal models (see [12], [11]). We also use standard notions and facts from the theory of algebraic varieties and schemes (see [5]). All algebraic varieties are assumed to be defined over $\mathbb{C}$. Throughout the paper we use standard notions and notation from [12], [11], [5]. However, let us introduce some:

- We denote by $\mathrm{Sing}(V)$ the singular locus of an algebraic variety V. For $P \in \mathrm{Sing}(V)$, we denote by $(O \in V)$ the analytic germ of V at P.
- For a $\mathbb{Q}$-Cartier divisor M and an algebraic cycle Z on a normal algebraic variety V, we denote by $M\big|_Z$ the restriction of M to Z. We denote by $Z_1 \cdot \ldots \cdot Z_k$ the intersection of algebraic cycles $Z_1, \ldots, Z_k$, $k \in \mathbb{N}$, in the Chow ring of V.
- $M_1 \equiv M_2$ (respectively, $Z_1 \equiv Z_2$) stands for the numerical equivalence of two $\mathbb{Q}$-Cartier divisors M_1, M_2 (respectively, of two algebraic 1-cycles Z_1, Z_2) on a normal algebraic variety V. We denote by $\rho(V)$ the Picard number of V. $D_1 \sim D_2$ stands for the the linear equivalence of two Weil divisors D_1, D_2 on V. We denote by $N_1(V)$ the group of classes of algebraic cycles on V modulo numerical equivalence. We denote by $\mathrm{Cl}(V)$ (respectively, $\mathrm{Pic}(V)$) the group of Weil (respectively, Cartier) divisors on V modulo linear equivalence.

- A normal algebraic three-dimensional variety V is called *Fano threefold* if it has at worst canonical Gorenstein singularities and the anticanonical divisor $-K_V$ is ample. A normal algebraic three-dimensional variety V is called *weak Fano threefold* if it has at worst canonical singularities and the anticanonical divisor $-K_V$ is nef and big. The number $(-K_V)^3$ (respectively, $\frac{1}{2}(-K_V)^3+1$) is called (anticanonical) *degree* (respectively, *genus*) of V.
- For a Weil divisor D on a normal algebraic variety V, we denote by $\mathcal{O}_V(D)$ the corresponding divisorial sheaf on V (sometimes we denote both by $\mathcal{O}_V(D)$ (or by D)).
- For a vector bundle $\mathcal{E}$ on smooth projective variety V, we denote by $c_i(\mathcal{E})$ the i-th Chern class of $\mathcal{E}$.
- We denote by $T_P(V)$ the Zariski tangent space to an algebraic variety V at a point $P \in V$. For V smooth and a smooth hypersurface $D \subset V$, we denote by $T_V\langle D\rangle$ the subsheaf of the tangent sheaf on V which consists of all vector fields tangent to D.
- For a Cartier divisor M on a normal projective variety V, we denote by $|M|$ the corresponding complete linear system on V. For an algebraic cycle Z on V, we denote by $|M - Z|$ the linear subsystem in $|M|$ which consists of all divisors passing through Z. For a linear system $\mathcal{M}$ on V without base components, we denote by $\Phi_{\mathcal{M}}$ the corresponding rational map.
- For a birational map $\psi : V' \dashrightarrow V$ between normal projective varieties and an algebraic cycle Z (respectively, a linear system $\mathcal{M}$) on V, we denote by $\psi_*^{-1}(Z)$ (respectively, by $\psi_*^{-1}(\mathcal{M})$) the proper transform of Z (respectively, of $\mathcal{M}$) on V'.
- We denote by $\mathbb{F}_n$ the Hirzebruch surface with the class of a fiber l and the minimal section h of the natural projection $\mathbb{F}_n \to \mathbb{P}^1$ such that $(h^2) = -n$, $n \in \mathbb{Z}_{\geqslant 0}$.

3. Preliminaries

In what follows, X is a Fano 3-fold of genus 36 (or degree 70). Let us present the construction and some properties of X (see [9] for more details).

Consider the weighted projective space $\mathbb{P} := \mathbb{P}(1,1,4,6)$ with weighted homogeneous coordinates x_0, x_1, x_2, x_3 of weights 1, 1, 4, 6, respectively. $\mathbb{P}$ is a Fano 3-fold of degree 72. Furthermore, the linear system $|-K_{\mathbb{P}}|$ gives an embedding of $\mathbb{P}$ in $\mathbb{P}^{38}$ such that the image $\Phi_{|-K_{\mathbb{P}}|}(\mathbb{P})$ is an intersection of quadrics. In what follows, we assume that $\mathbb{P} \subset \mathbb{P}^{38}$ is anticanonically embedded. Then $L := \mathrm{Sing}(\mathbb{P})$ is a line on $\mathbb{P}$ with respect to this embedding. Moreover, there are two points P and Q on L such that the singularities $P \in \mathbb{P}$, $Q \in \mathbb{P}$ are of types $\frac{1}{6}(4,1,1)$, $\frac{1}{4}(2,1,1)$, respectively, and for every point $O \in L \setminus \{P,\ Q\}$ the singularity $O \in \mathbb{P}$ is analytically isomorphic to $(0, o) \in \mathbb{C} \times W$, where $o \in W$ is the singularity of type $\frac{1}{2}(1,1)$ (see [9, Example 2.13]).

Proposition 3.1. *L is the unique line on $\mathbb{P}$.*

Proof. Let $L_0 \neq L$ be another line on $\mathbb{P}$. Since $-K_{\mathbb{P}} \sim \mathcal{O}_{\mathbb{P}}(12)$, we have

$$\mathcal{O}_{\mathbb{P}}(1) \cdot L_0 = \frac{1}{12}, \tag{3.2}$$

which implies that $L \cap L_0 \neq \emptyset$. Consider the crepant resolution $\phi : T \longrightarrow \mathbb{P}$ of $\mathbb{P}$. Set $L_0' := \phi_*^{-1}(L_0)$, $E_Q := \phi^{-1}(Q)$, $E_P := \phi^{-1}(P)$ and $E_L := \overline{\phi^{-1}(L \setminus \{P,Q\})}$, the Zariski closure in T of $\phi^{-1}(L\setminus\{P,Q\})$. These are all the components of the ϕ-exceptional locus. Furthermore, we have $E_P = E_P^{(1)} \cup E_P^{(2)}$, where $E_P^{(i)}$ are irreducible components of the divisor E_P such that $E_P^{(1)} \cap E_L = \emptyset$ and $E_P^{(2)} \cap E_L \neq \emptyset$ (see [9, Example 2.13] for the explicit construction of ϕ).

Since $\rho(\mathbb{P}) = 1$, the group $N_1(T)$ is generated by the classes of ϕ-exceptional curves and some curve Z on T such that $R := \mathbb{R}_+[Z]$ is the K_T-negative extremal ray (see [24, Lemmas 4.2, 4.3]). In particular, since $-K_T \cdot L_0' = 1$, [24, Lemmas 4.2, 4.3] implies that

$$L_0' \equiv Z + E^*, \tag{3.3}$$

where E^* is a linear combination with nonnegative coefficients of irreducible ϕ-exceptional curves. Further, the linear projection π_L of $\mathbb{P}$ from L is given by the linear system $\mathcal{H} \subset |-K_{\mathbb{P}}|$ of all hyperplane sections of $\mathbb{P}$ containing L. In addition, π_L maps L_0 to the point because $L \cap L_0 \neq \emptyset$ and $\mathbb{P}$ is the intersection of quadrics. On the other hand, ϕ factors through the blow up of $\mathbb{P}$ at L (see [9], [8]). Hence the linear system $\phi_*^{-1}\mathcal{H}$ is basepoint-free on T and $H \cdot L_0' = 0$ for $H \in \phi_*^{-1}\mathcal{H}$. In particular, $H \in |-K_T - E_L|$.

Lemma 3.4. *In* (3.3)*, the support* $\mathrm{Supp}(E^*)$ *of* E^* *is either* $\emptyset$ *or* e_P*, where* $e_P \subset E_P^{(1)}$.

Proof. As we saw, the face of the Mori cone $\overline{NE}(T)$, which corresponds to the nef divisor H, contains the class of the curve L_0'. Then from (3.3) we get

$$H \cdot Z = H \cdot E^* = 0.$$

In particular, H intersects trivially every curve in $\mathrm{Supp}(E^*)$. On the other hand, we have $\mathrm{Supp}(E^*) \subseteq \{e_P, e_Q, e_L\}$, where e_P, e_Q, e_L are the curves in E_P, E_Q, E_L, respectively. But for $e_P \subset E_P^{(2)}$ intersections $H \cdot e_P$, $H \cdot e_Q$, $H \cdot e_L$ are all non-zero. Thus, $\mathrm{Supp}(E^*)$ is either $\emptyset$ or e_P, where $e_P \subset E_P^{(1)}$. ☐

Consider the extremal contraction $f_R : T \longrightarrow T'$ of R. The morphism f_R is birational with the exceptional divisor E_R (see [9], [8]).

Lemma 3.5. *The divisor* $-K_{T'}$ *is not nef.*

Proof. Suppose that $-K_{T'}$ is nef, i.e., T' is a weak Fano 3-fold (with possibly non-Gorenstein singularities). If T' has only terminal factorial singularities, then since $(-K_{T'})^3 \geqslant (-K_T)^3 = 72$ (see [24, Proposition-definition 4.5]), T' is a terminal $\mathbb{Q}$-factorial modification either of $\mathbb{P}(1,1,1,3)$ or of $\mathbb{P}(1,1,4,6)$. In particular, either $\rho(Y') = 5$ or $\rho(Y') = 2$ (see [9], [8]). On the other hand, $\rho(T') = \rho(T) - 1 = 4$, a contradiction.

Thus, the singularities of T' are worse than factorial. In this case, $f_R(E_R)$ is a point (see [24, Proposition-definition 4.5]) and we get

$$E_P \cap E_R = E_Q \cap E_R = \emptyset. \tag{3.6}$$

On the other hand, it follows from (3.3) that $-K_{\mathbb{P}} \cdot \phi_*(Z) = 1$, i.e., $\phi(Z)$ is a line on $\mathbb{P}$. In particular, as for L_0 above, we have $\phi_*(Z) \cap L \neq \emptyset$. But then (3.6) implies that $0 = K_T \cdot Z = -1$, a contradiction. ☐

It follows from Lemma 3.5 that $E_R = \mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$ (see [24, Proposition-definition 4.5]). But if $E_R = \mathbb{F}_1$, then $\phi(E_R)$ is a plane on $\mathbb{P}$ such that $L \not\subset \phi(E_R)$ (see [24, Proposition-definition 4.5]). This implies that there is a line on $\mathbb{P}$ not intersecting L, a contradiction (see (3.2)). Finally, in the case when $E_R = \mathbb{P}^1 \times \mathbb{P}^1$, we have $Z \subset E_R = E_L$ (see [24, Proposition-definition 4.5]), and if $\mathrm{Supp}(E^*) = \emptyset$ in (3.3), then $L_0 = L$, a contradiction. Hence, by Lemma 3.4, we get $\mathrm{Supp}(E^*) = e_P$, where $e_P \subset E_P^{(1)}$. Further, on E_R we have:

$$Z \sim l, \qquad E_P\big|_{E_R} = E_P^{(2)}\big|_{E_R} \sim h \sim E_Q\big|_{E_R},$$

which implies that $E_P^{(2)} \cdot Z = E_Q \cdot Z = 1$. On the other hand, since $L_0 \neq L$, we have either $E_P^{(2)} \cdot L_0' = 0$ or $E_Q \cdot L_0' = 0$. Then, intersecting (3.3) with $E_P^{(2)}$ and E_Q, we get a contradiction because $E_P^{(2)} \cdot e_P$ and $E_Q \cdot e_P \geqslant 0$.

Thus, we get $L_0 = L$, a contradiction. Proposition 3.1 is completely proved. ☐

Coming back to the construction of X, take any point O in $L \setminus \{P,\ Q\}$ and consider the linear projection $\pi : \mathbb{P} \dashrightarrow \mathbb{P}^{37}$ from O. Then the image of π is a Fano 3-fold X_O of degree 70 (see [9], [8]).

Proposition 3.7. *For any point* O' *in* $L \setminus \{P,\ Q,\ O\}$*, the image of the linear projection* $\mathbb{P} \dashrightarrow \mathbb{P}^{37}$ *from* O' *is a Fano* 3*-fold* $X_{O'}$ *isomorphic to* X_O*.*

Proof. In the above notation, L is given by equations $x_0 = x_1 = 0$ on $\mathbb{P}$, with equations of P and Q being $x_0 = x_1 = x_2 = 0$ and $x_0 = x_1 = x_3 = 0$, respectively (see [6, 5.15]). Then the torus $(\mathbb{C}^*)^3$, acting on $\mathbb{P}$, acts transitively on the set $L \setminus \{P,\ Q\}$, which induces an isomorphism $X_{O'} \simeq X_O$. ☐

In what follows, because of Proposition 3.7, we fix the point $O \in L \setminus \{P,\ Q\}$, the linear projection $\pi : \mathbb{P} \dashrightarrow \mathbb{P}^{37}$ from O, and denote the image of π by X. Let us construct a terminal $\mathbb{Q}$-factorial modification of X. Consider the blow up $\sigma : W \longrightarrow \mathbb{P}$ of $\mathbb{P}$ at O, and the following commutative diagram:

$$\begin{tikzcd} & W \arrow[dl, "\sigma"'] \arrow[dr, "\mu"] & \\ \mathbb{P} \arrow[rr, dashed, "\pi"'] & & X. \end{tikzcd}$$

The type of the singularity $O \in \mathbb{P}$ implies that W has at most canonical Gorenstein singularities. Moreover, we have $\mathrm{Sing}(W) = \sigma_*^{-1}(L)$ and the singularities of W are exactly of the same kind as of $\mathbb{P}$, i.e., locally near every point

in $\mathrm{Sing}(W)$, W is isomorphic to $\mathbb{P}$. Then, resolving the singularities of W in the same way as for $\mathbb{P}$, we arrive at the birational morphism $\tau : Y \longrightarrow W$, with Y being smooth and $K_Y = \tau^*(K_W)$ (see [9], [8]). Set $f := \tau \circ \mu : Y \longrightarrow X$.

Proposition 3.8. *$f : Y \longrightarrow X$ is a terminal $\mathbb{Q}$-factorial modification of X. Moreover, Y is unique up to isomorphism, i.e., every smooth weak Fano 3-fold of degree 70 is isomorphic to Y.*

Proof. The linear projection π is given by the linear system $\mathcal{H} \subset |-K_{\mathbb{P}}|$ of all hyperplane sections of $\mathbb{P}$ passing through O. For a general $H \in \mathcal{H}$, we have

$$\sigma_*^{-1}(H) = \sigma^*(H) - E_\sigma,$$

where E_σ is the σ-exceptional divisor. On the other hand, from the adjunction formula we get

$$K_W = \sigma^*(K_{\mathbb{P}}) + E_\sigma.$$

Thus, the morphism $\mu : W \longrightarrow X$ is given by the linear system $\sigma_*^{-1}(\mathcal{H}) \subseteq |-K_W|$. Furthermore, since $\mathbb{P}$ is an intersection of quadrics, π is a birational map, which implies that μ and f are also birational with $K_Y = f^*(K_X)$. In particular, $(-K_Y)^3 = (-K_X)^3 = 70$.

Thus, it remains to prove that every smooth weak Fano 3-fold of degree 70 is isomorphic to Y. Let Y' be another smooth weak Fano 3-fold of degree 70. Then its image under the morphism $f' := \Phi_{|-nK_{Y'}|}$, $n \in \mathbb{N}$, is a Fano threefold X' such that $K_{Y'} \equiv f'^*(K_{X'})$ (see [10]). Since $(-K_{Y'})^3 = (-K_{X'})^3 = 70$, we get $X' \simeq X$ and Y' is a terminal $\mathbb{Q}$-factorial modification of X. Then, since Y and Y' are relative minimal models over X, the induced birational map $Y \dashrightarrow Y'$ is either an isomorphism or a sequence of K_Y-flops over X (see [12]).

Lemma 3.9. *Every K_Y-trivial extremal birational contraction $f_1 : Y \longrightarrow Y_1$ is divisorial.*

Proof. Suppose that f_1 is small. In the notation from the proof of Proposition 3.1, denote by $E_{Y,L}$, $E_{P,L}^{(i)}$, $E_{Q,L}$ the proper transforms on Y of surfaces E_L, $E_P^{(i)}$, E_Q, respectively. The resolution $\tau : Y \longrightarrow W$ (or $\phi : T \longrightarrow \mathbb{P}$) is locally toric near $\mathrm{Sing}(W)$. In particular, we have $E_{P,L}^{(1)} \simeq \mathbb{F}_4$, $E_{P,L}^{(2)} \simeq \mathbb{F}_2$, $E_{Q,L} \simeq \mathbb{F}_2$, $E_{Y,L} \simeq \mathbb{F}_m$ for some $m \in \mathbb{N}$ (see [9, Example 2.13]), and hence the only possibility for f_1 is to contract the curve $Z = h$ on $E_{Y,L}$ such that $\tau(Z) = \sigma_*^{-1}(L)$.

On the other hand, Y is obtained by the blow up of the 3-fold T at the curve $\phi^{-1}(O) \simeq \mathbb{P}^1$ (see [9], [8]). Furthermore, since $\mathbb{P}$ is singular along the line, we have $E_L \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (see [24, Proposition-definition 4.5]), and hence $E_{Y,L} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, a contradiction. ☐

It follows from Lemma 3.9 that $Y' \simeq Y$. Proposition 3.8 is completely proved. ☐

Corollary 3.10. $\mathrm{Sing}(X)$ *consists of a unique point.*

Proof. Since the morphism $\mu : W \longrightarrow X$ is given by the linear system $\sigma_*^{-1}(\mathcal{H}) \subseteq |-K_W| = |\sigma^*(-K_{\mathbb{P}}) - E_\sigma|$ (see the proof of Proposition 3.8), it contracts only $\sigma_*^{-1}(L) = \mathrm{Sing}(W)$ to the unique singular point on X (see Proposition 3.1). ☐

Corollary 3.11. *We have* $\mathrm{Pic}(X) = \mathbb{Z} \cdot K_X$ *and* $\mathrm{Cl}(X) = \mathbb{Z} \cdot K_X \oplus \mathbb{Z} \cdot E$, *where* $E := \mu_*(E_\sigma)$.

Proof. This follows from the construction of X and equalities $\rho(\mathbb{P}) = 1$, $(-K_X)^3 = 70$. ☐

Remark 3.12. It follows from the construction of X that $f = \Phi_{|-K_Y|}$ and $X \subseteq \mathbb{P}^{37}$ is anticanonically embedded.

Remark 3.13. Since Y is a smooth weak Fano 3-fold, we have $\mathrm{Pic}(Y) \simeq H^2(Y, \mathbb{Z})$ (see [7, Proposition 2.1.2]) and $H^2(Y, \mathcal{O}_Y) = 0$ by Kawamata–Viehweg Vanishing Theorem.

It follows from Corollary 3.10 that a general surface $S \in |-K_X|$ is smooth. Furthermore, Corollary 3.11 implies that the cycles $K_X\big|_S$ and $E\big|_S$ are not divisible in $\mathrm{Pic}(S)$, linearly independent in $H^2(S, \mathbb{Q})$, and hence they generate a primite sublattice R_S in $\mathrm{Pic}(S)$. It follows from the construction of X that all lattices R_S, $S \in |-K_X|$, are isomorphic to the lattice $\mathrm{R} \simeq \mathbb{Z}^2$ with the associated quadratic form $70x^2 + 4xy - 2y^2$, and we can consider the moduli stack $\mathcal{K} := \mathcal{K}_{36}^{\mathrm{R}}$ of K3 surfaces of type R (see [1, (2.3)]). $\mathcal{K}$ is actually an algebraic space because the forgetful map $\mathcal{K} \longrightarrow \mathcal{K}_{36}$ is representable and 1-to-1 in our case (see [1, (2.5)]).[1)]

Proposition 3.14 (see [1])**.** *Let S be the* K3 *surface of type* R*. Then*

[1)]It can be also easily seen that the class of a (-2)-curve in $\mathrm{Pic}(S)$ is unique and generated by the conic $E|_S$.

- *the first order deformations of* (S, R) *are parameterized by the orthogonal of* $c_1(\mathrm{R}) \subset H^1(S, \Omega^1_S)$ *in* $H^1(S, T_S)$*;*
- *the space* $\mathcal{K}$ *is smooth, irreducible, of dimension* 18*.*

4. Proof of Theorem 1.2

We use the notation and conventions of Section 3. Since $f : Y \longrightarrow X$ is the crepant resolution (see Proposition 3.8), it follows from Corollary 3.10 that we can assume a general $S \in |-K_X|$ to be a surface in $|-K_Y|$ on Y. We can also assume that $S \cap \mathrm{Exc}(f) = \emptyset$ for the f-exceptional locus $\mathrm{Exc}(f)$. Further, it follows from Remark 3.12 that the points in $(\mathbb{P}^{37})^*$, corresponding to such S's, form an open subset $U \subset (\mathbb{P}^{37})^*$. Consider the natural (faithful) action of the group $G := \mathrm{Aut}(Y)$ on U. Shrinking U if necessary, we obtain the following

Proposition 4.1. *The geometric quotient U/G exists as a smooth scheme.*

Proof. Let us calculate the group G first. Take $g \in \mathrm{Aut}(\mathbb{P})$ to be an automorphism of $\mathbb{P}$ which fixes the point O. Then g lifts to the automorphism of Y (see the construction of X and Y in Section 3). Conversely, take any $g \in G$.

Lemma 4.2. *The morphism $\tau : Y \longrightarrow W$ is g-equivariant.*

Proof. Since the morphism $f = \Phi_{|-K_Y|} : Y \longrightarrow X$ is g-equivariant (see Remark 3.12), it follows from the construction of Y in Section 3 that the irreducible components of $\mathrm{Exc}(f)$ are all g-invariant. Thus, since $\mathrm{Pic}(Y)$ is generated by K_Y, the irreducible components of E_f and $E_{Y,\sigma} := \tau_*^{-1}(E_\sigma)$, it is enough to prove that $g(E_{Y,\sigma}) = E_{Y,\sigma}$. Suppose that $g(E_{Y,\sigma}) \neq E_{Y,\sigma}$. Then, since all the curves in E_σ (respectively, in $\tau_*(g(E_{Y,\sigma}))$) are numerically proportional and τ is divisorial, we must have $E_\sigma \cap \tau_*(g(E_{Y,\sigma})) = \emptyset$. The latter implies that there exists a curve $C \equiv \sigma_*(-K_W \cdot \tau_*(g(E_{Y,\sigma})))$ on $\mathbb{P}$ with $-K_{\mathbb{P}} \cdot C = 4$ and $C \cap L = \emptyset$. On the other hand, since $-K_{\mathbb{P}} \sim \mathcal{O}_{\mathbb{P}}(12)$, we get $\mathcal{O}_{\mathbb{P}}(1) \cdot C = \dfrac{1}{3}$, a contradiction. □

It follows from Lemma 4.2 that g acts on W. Further, considering the induced g-action on the cone $\overline{NE}(W)$, we obtain, since $\mathrm{Pic}(W) = \mathbb{Z} \cdot K_W \oplus \mathbb{Z} \cdot E_\sigma$, that $\sigma : W \longrightarrow \mathbb{P}$ is g-equivariant. The latter gives a g-action on $\mathbb{P}$ with the fixed point O.

Thus, G is isomorphic to the stabilizer in $\mathrm{Aut}(\mathbb{P})$ of the point O, and to describe the G-action on U we may consider the action of the corresponding subgroup in $\mathrm{Aut}(\mathbb{P})$ on the linear system $|-K_{\mathbb{P}} - O|$. Note that, since $P \in \mathbb{P}$, $Q \in \mathbb{P}$, $O \in \mathbb{P}$ are the pairwise non-isomorphic singularities, every $g \in G$ fixes every point on L. Finally, since $\mathcal{O}_{\mathbb{P}}(1)$, $\mathcal{O}_{\mathbb{P}}(4)$, $\mathcal{O}_{\mathbb{P}}(6)$ are G-invariant, the g-action on $\mathbb{P}$ can be described as follows:

$$
\begin{aligned}
x_0 &\mapsto a x_0 + b x_1,\\
x_1 &\mapsto c x_0 + d x_1,\\
x_2 &\mapsto \lambda^4 x_2 + f_4(x_0, x_1),\\
x_3 &\mapsto \lambda^6 x_3 + x_2 f_2(x_0, x_1) + f_6(x_0, x_1),
\end{aligned}
\tag{4.3}
$$

where $\lambda \in \mathbb{C}^*$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{C})/\{\pm 1\}$, $f_i := f_i(x_0, x_1)$ are arbitrary homogeneous polynomials of degree i in x_0, x_1. On the other hand, since $-K_{\mathbb{P}} \sim \mathcal{O}_{\mathbb{P}}(12)$, a general element in $|-K_{\mathbb{P}} - O|$ can be given by the equation

$$
\alpha x_3^2 + x_2^3 + a_6(x_0, x_1) x_3 + a_2(x_0, x_1) x_2 x_3 + a_4(x_0, x_1) x_2^2 + a_8(x_0, x_1) x_2 + a_{12}(x_0, x_1) = 0 \tag{4.4}
$$

on $\mathbb{P}$, where $a_i := a_i(x_0, x_1)$ are arbitrary general homogeneous polynomials in x_0, x_1 of degree i, and $\alpha \in \mathbb{C}^*$ is fixed.

Take a general surface S_0 on $\mathbb{P}$ with the equation (4.4) such that $a_2 = a_4 = a_6 = 0$.

Lemma 4.5. *If S_0 is g-invariant for some $g \neq \mathrm{id}$ from (4.3), then $f_2 = f_4 = f_6 = 0$, $c = b = 0$, $a = d = \sqrt{-1}$, $\lambda^4 = 1$.*

Proof. g-invariance of S_0 implies that $f_2 = f_4 = f_6 = 0$ and

$$
a_8(x_0, x_1) = a_8(a x_0 + b x_1, c x_0 + d x_1), \qquad a_{12}(x_0, x_1) = a_{12}(a x_0 + b x_1, c x_0 + d x_1). \tag{4.6}
$$

Without loss of generality we may assume that $a_8 = x_0 x_1 b_6$ for some $b_6 := b_6(x_0, x_1)$ coprime to x_0 and x_1. Then (4.6) and generality of S_0 imply that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$, and we get:

$$
a^{12} = 1, \qquad a^{i+1} d^{7-i} = 1, \qquad a^i d^{6-i} = a^j d^{6-j}
$$

for all $0 \leqslant i,\ j \leqslant 6$. In particular, $a = d$, $a^8 = a^{12} = 1$, i.e., $a = d = \sqrt{-1}$. Finally, since $x_2 \mapsto \lambda^4 x_2$ (see (4.3)) and hence $a_8(x_0, x_1) = \lambda^4 a_8(x_0, x_1)$ (see (4.4)), we get $\lambda^4 = 1$. $\square$

Lemma 4.7. *Let $g \in G$, given by (4.3), be such that $f_2 = f_4 = f_6 = 0$, $c = b = 0$, $a = d = \sqrt{-1}$, $\lambda = \pm\sqrt{-1}$. Then $g = \mathrm{id}$.*

Proof. We have

$$g([x_0 : x_1 : x_2 : x_3]) = [\sqrt{-1}x_0 : \sqrt{-1}x_1 : (\sqrt{-1})^4 x_2 : (\sqrt{-1})^6 x_3] = [x_0 : x_1 : x_2 : x_3]$$

on $\mathbb{P}$. Hence $g = \mathrm{id}$. $\square$

It follows from Lemmas 4.5 and 4.7, since $\lambda^4 = 1$ implies $\lambda^2 = \pm 1$, that the stabilizer of S_0 in G is a group of order 2, generated by some $g_0 \in G$ with $\lambda^2 = 1$ (see (4.3)). Consider the normal algebraic subgroup $G' \subset G$ generated by $g^{-1} g_0 g$ for all $g \in G$, i.e., generators of G' are all the elements in G for which $f_4 = 0$, $c = b = 0$, $a = d = \sqrt{-1}$ and $\lambda = 1$ in (4.3). Then the G'-action on U is proper, and we can consider the geometric quotient $U' := U/G'$, which exists as a normal scheme (see [22]). Further, take the $G'' := G/G'$-equivariant factorization map $\pi_G : U \longrightarrow U'$ and consider the induced G''-action on U'. Shrinking U if necessary, we obtain

Lemma 4.8. *The G''-action on U' is free.*

Proof. Let S_0' be the image on U' of S_0 under π_G. Then we have $G'' \cdot S_0' \simeq G''$ for the G''-orbit of S_0', and, by the dimension count, there exists a Zariski open subset in U' with a free G''-action. $\square$

Lemma 4.8 and [22] imply that the geometric quotient $U/G \simeq U'/G''$ exists as a smooth scheme. Proposition 4.1 is completely proved. $\square$

Set $\mathcal{F} := U/G$ to be the scheme from Proposition 4.1. It follows from Proposition 3.8 and Remark 3.12 that $\mathcal{F}$ is a (coarse) moduli space which parameterizes the pairs $(Y^\sharp, S^\sharp)$ consisting of smooth weak Fano 3-fold $Y^\sharp$ of degree 70 and smooth surface $S^\sharp \in |-K_{Y^\sharp}|$ (cf. [1, (2.2)]). These give the following

Lemma 4.9. *For $o := (Y, S) \in \mathcal{F}$, we have $H^1(Y, T_Y\langle S\rangle) = T_o\mathcal{F}$.*

Proof. This follows from the fact that $\mathcal{F}$ is smooth and $H^1(Y, T_Y\langle S\rangle)$ parameterizes the first order deformations of (Y, S) (see [1, Proposition 1.1]). $\square$

Consider the forgetful morphism $s : \mathcal{F} \longrightarrow \mathcal{K}$, which sends (Y, S) to S.

Lemma 4.10. *s is generically surjective.*

Proof. Consider the restriction map $r : T_Y\langle S\rangle \longrightarrow T_S$. It fits into the exact sequence

$$0 \longrightarrow \Omega_Y^2 \longrightarrow T_Y\langle S\rangle \xrightarrow{r} T_S \longrightarrow 0, \tag{4.11}$$

since $\mathrm{Ker}(r) = T_Y(-S)$ is a subsheaf of $T_Y\langle S\rangle$ consisting of the vector fields vanishing along S, for which we have $T_Y(-S) \simeq \Omega_Y^2$. From (4.11) we get the exact sequence

$$H^1(Y, T_Y\langle S\rangle) \xrightarrow{H^1(r)} H^1(S, T_S) \xrightarrow{\partial} H^2(Y, \Omega_Y^2).$$

The map ∂ is dual to the restriction map $i : H^1(Y, \Omega_Y^1) \longrightarrow H^1(S, \Omega_S^1)$ (see [1]). In particular, $\mathrm{Ker}(\partial)$ is the orthogonal of $\mathrm{Im}(i)$. On the other hand, we have $\mathrm{Im}(i) = \mathbb{Z} \cdot c_1(K_Y\big|_S) \oplus \mathbb{Z} \cdot c_1(\tau_*^{-1}(E_\sigma)\big|_S) \simeq \mathbb{Z} \cdot K_X\big|_S \oplus \mathbb{Z} \cdot E\big|_S$ (see Corollary 3.11 and Remark 3.13), and hence $H^1(r)$ coincides with the tangent map to s at (Y, S), with $\mathrm{Im}(H^1(r)) = \mathrm{Ker}(\partial)$ being the tangent space to $\mathcal{K}$ at S (see Lemma 4.9 and Proposition 3.14). Thus, since $\mathcal{K}$ is irreducible (see Proposition 3.14), we get that s is generically surjective. $\square$

Theorem 1.2 is completely proved.

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